

ON BASES OF QUANTIZED ENVELOPING ALGEBRAS

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ABSTRACT. We give a systematic description of many monomial bases for a given quantized enveloping algebra and of many integral monomial bases for the associated Lusztig $\mathbb{Z}[v, v^{-1}]$ -form. The relations between monomial bases, PBW bases and canonical bases are also discussed.

1. INTRODUCTION

Let \mathfrak{g} be a (complex) semisimple Lie algebra and let \mathbf{U}^+ be the positive part of its associated quantized enveloping algebra $\mathbf{U} = \mathbf{U}_v(\mathfrak{g})$ over $\mathbb{Q}(v)$ with a Drinfeld-Jimbo presentation in the generators $E_i, F_i, K_i^{\pm 1}$ ($i \in I = [1, n]$). We denote by U^+ the Lusztig form of \mathbf{U}^+ , that is, U^+ is generated by all the divided powers $E_i^{(m)}$ over $\mathcal{Z} := \mathbb{Z}[v, v^{-1}]$. Let Ω be the set of words on the alphabet I and, for $w = i_1^{e_1} i_2^{e_2} \cdots i_m^{e_m} \in \Omega$ with $i_{j-1} \neq i_j \forall j$, put $\mathbf{m}^{(w)} = E_{i_1}^{(e_1)} \cdots E_{i_m}^{(e_m)}$. Let further Λ denote the set of all functions from the set of positive roots of \mathfrak{g} to non-negative integers. The main result of the paper is the following.

Theorem 1.1. *Assume that \mathfrak{g} is simply-laced. Then there is a partition $\Omega = \cup_{\lambda \in \Lambda} \Omega_\lambda$ such that, for any chosen $w_\lambda \in \Omega_\lambda$ ($\lambda \in \Lambda$), the set of monomials $\{\mathbf{m}^{(w_\lambda)}\}_{\lambda \in \Lambda}$ forms a basis for \mathbf{U}^+ . Moreover, if all words w_λ are chosen to be distinguished, then the set forms a \mathcal{Z} -basis for U^+ .*

This work generalizes some constructions of monomial bases given in [15] and [11]; see [3] for a similar result in the affine \mathfrak{sl}_n case. The assumption of simply-laced types is made so that we may directly use the theory of quiver representations, especially the theory of generic extensions developed recently in [10]. It is natural to expect that a similar result holds in the non-simply-laced case.

The main ingredients for the proof are Ringel's Hall algebra theory, Reineke's monoidal structure on the set \mathcal{M} of isoclasses of finite dimensional representations of a Dynkin quiver Q and the Bruhat-Chevalley type partial ordering on \mathcal{M} . These will be discussed separately in §2, §3 and §4. Distinguished words are introduced and investigated in §5 and we shall prove the main result in §6. As an application of the theory, we mention an elementary construction (see [10, §6]) of the canonical bases for U^+ as the counterpart of a similar construction for Hecke algebra in [7]. This construction uses the same order as the one used in the geometric construction which involves perverse sheaf and intersection cohomology theories. Finally, more explicit results on distinguished words are worked out for the case of type A in the last section.

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Throughout, k denotes a finite field unless otherwise specified. Let $q_k = |k|$. All modules are finite dimensional over k . If M is a module, nM , $n \geq 0$, denotes the direct sum of n copies of M . Further, by $[M]$ we denote the class of modules isomorphic to M , i.e., the isoclass of M . For modules M, N_1, \dots, N_t , let $F_{N_1 \dots N_t}^M$ denote the number of filtrations

$$M = M_0 \supset M_1 \supset \dots \supset M_{t-1} \supset M_t = 0$$

such that $M_{i-1}/M_i \cong N_i$ for all $1 \leq i \leq t$.

2. RINGEL-HALL ALGEBRAS OF DYNKIN QUIVERS

Let $Q = (I, Q_1)$ be a quiver, i.e., a finite directed graph, where $I = Q_0$ is the set of vertices $\{1, 2, \dots, n\}$ and Q_1 is the set of arrows. If $\rho \in Q_1$ is an arrow from tail i to head j , we write $h(\rho)$ for j and $t(\rho)$ for i . Thus we obtain functions $h, t : Q_1 \rightarrow I$. A vertex $i \in I$ is called a sink (resp. source) if there is no arrow ρ with $t(\rho) = i$ (resp. $h(\rho) = i$).

Let kQ be the path algebra of Q . A (finite dimensional) *representation* V of Q , consisting of a set of finite dimensional vector spaces V_i for each $i \in I$ and a set of linear transformations $V_\rho : V_{t(\rho)} \rightarrow V_{h(\rho)}$ for each $\rho \in Q_1$, is identified with a (left) kQ -module. We call $\mathbf{dim} V := (\dim V_1, \dots, \dim V_n)$ the *dimension vector* of V and $\ell(V) := \sum_{i=1}^n \dim V_i$ the *length*¹ of V . In case Q contains no oriented cycles, there are exactly n pairwise non-isomorphic simple kQ -modules S_1, \dots, S_n corresponding bijectively to the vertices of Q .

From now on, we assume that Q is a *Dynkin quiver*, that is, a quiver whose underlying graph is a (simply laced) Dynkin graph. By Gabriel's theorem [5], there is a bijection between the set of isoclasses of indecomposable kQ -modules and a positive system Φ^+ of the root system Φ associated with Q . For any $\beta \in \Phi^+$, let $M(\beta) = M_k(\beta)$ denote the corresponding indecomposable kQ -module. By the Krull-Remak-Schmidt theorem, every kQ -module M is isomorphic to

$$M(\lambda) = M_k(\lambda) := \bigoplus_{\beta \in \Phi^+} \lambda(\beta) M_k(\beta),$$

for some function $\lambda : \Phi^+ \rightarrow \mathbb{N}$. Thus, the isoclasses of kQ -modules are indexed by the set

$$\Lambda = \{\lambda : \Phi^+ \rightarrow \mathbb{N}\} \cong \mathbb{N}^{|\Phi^+|}.$$

Further, by a result of Ringel [12], for $\lambda, \mu_1, \dots, \mu_m$ in Λ , there is a polynomial $\varphi_{\mu_1 \dots \mu_m}^\lambda(q) \in \mathbb{Z}[q]$, called Hall polynomial, such that for any finite field k of q_k elements

$$\varphi_{\mu_1 \dots \mu_m}^\lambda(q_k) = F_{M_k(\mu_1) \dots M_k(\mu_m)}^{M_k(\lambda)}.$$

Let $\mathcal{A} = \mathbb{Z}[q]$ be the integral polynomial ring in the indeterminate q . The generic (untwisted) Ringel-Hall algebra $\mathcal{H} = \mathcal{H}_q(Q)$ of Q over \mathcal{A} is by definition the free \mathcal{A} -module having basis $\{u_\lambda | \lambda \in \Lambda\}$, and satisfying the multiplicative relations:

$$u_\mu u_\nu = \sum_{\lambda \in \Lambda} \varphi_{\mu\nu}^\lambda(q) u_\lambda.$$

We sometimes write $u_\lambda = u_{[M(\lambda)]}$ in order to make certain calculations in term of modules. For $i \in I$, we set $u_i = u_{[S_i]}$. Clearly, \mathcal{H} admits a natural \mathbb{N}^n -grading by dimension vectors.

¹This is also called the dimension of V .

Following [14], we can twist the multiplication of the Ringel-Hall algebra to obtain the positive part \mathbf{U}^+ of a quantized enveloping algebra.

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$, where v is an indeterminate with $v^2 = q$. The *twisted* Ringel-Hall algebra $\mathcal{H}^* = \mathcal{H}_v^*(Q)$ of Q is by definition the free \mathcal{Z} -module having basis $\{u_\lambda = u_{[M(\lambda)]} | \lambda \in \Lambda\}$ and satisfying the multiplication rules

$$u_\mu \star u_\nu = v^{\langle \mu, \nu \rangle} u_\mu u_\nu = v^{\langle \mu, \nu \rangle} \sum_{\lambda \in \Lambda} \varphi_{\mu\nu}^\lambda(v^2) u_\lambda,$$

where $\langle \mu, \nu \rangle = \dim_k \operatorname{Hom}_k Q(M(\mu), N(\nu)) - \dim_k \operatorname{Ext}_k^1(M(\mu), N(\nu))$ is the Euler form associated with the quiver Q . Note that, if we define the bilinear form $\langle -, - \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$ by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i \in I} a_i b_i - \sum_{\rho \in Q_1} a_{t(\rho)} b_{h(\rho)},$$

where $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$, then

$$\langle \mu, \nu \rangle = \langle \dim M(\mu), \dim M(\nu) \rangle.$$

For each $m \geq 1$, let $[m] = \frac{v^m - v^{-m}}{v - v^{-1}}$ and $[m]! = [1][2] \cdots [m]$. We define, for each $i \in I$, the divided powers

$$u_i^{(\star m)} = \frac{u_i^{\star m}}{[m]!} \quad \text{and} \quad E_i^{(m)} = \frac{E_i^m}{[m]!},$$

in \mathcal{H}^* and \mathbf{U}^+ , respectively. Here $u_i^{\star m} = \underbrace{u_i \star \cdots \star u_i}_m = v^{\binom{m}{2}} u_i^m$.

The following result is due to Ringel [15, §7].

Proposition 2.1. *The algebra \mathcal{H}^* is generated by all $u_i^{(\star m)}$ ($i \in I, m \geq 1$). Moreover, there is a natural isomorphism*

$$\Psi : \mathbf{U}^+ \xrightarrow{\sim} \mathcal{H}^*, \quad E_i^{(m)} \mapsto u_i^{(\star m)}, \quad (i \in I, m \geq 1).$$

In the sequel, we shall identify \mathbf{U}^+ with \mathcal{H}^* under this isomorphism.

3. GENERIC EXTENSIONS AND THE MONOID \mathcal{M}

In this section, we collect some recent results on generic extensions for quiver representations over an *algebraically closed* field k .

Fix $\mathbf{d} = (d_i)_i \in \mathbb{N}^n$ and define the affine space

$$R(\mathbf{d}) = R(Q, \mathbf{d}) := \prod_{\alpha \in Q_1} \operatorname{Hom}_k(k^{d_{t(\alpha)}}, k^{d_{h(\alpha)}}) \cong \prod_{\alpha \in Q_1} k^{d_{h(\alpha)} \times d_{t(\alpha)}}.$$

Thus, a point $x = (x_\alpha)_\alpha$ of $R(\mathbf{d})$ determines a representation $V(x)$ of Q . The algebraic group $GL(\mathbf{d}) = \prod_{i=1}^n GL_{d_i}(k)$ acts on $R(\mathbf{d})$ by conjugation

$$(g_i)_i \cdot (x_\alpha)_\alpha = (g_{h(\alpha)} x_\alpha g_{t(\alpha)}^{-1})_\alpha,$$

and the $GL(\mathbf{d})$ -orbits \mathcal{O}_x in $R(\mathbf{d})$ correspond bijectively to the isoclasses $[V(x)]$ of representations of Q with dimension vector \mathbf{d} .

The stabilizer $GL(\mathbf{d})_x = \{g \in GL(\mathbf{d}) | gx = x\}$ of x is the group of automorphisms of $M := V(x)$ which is Zariski-open in $\text{End}_{kQ}(M)$ and has dimension equal to $\dim \text{End}_{kQ}(M)$. It follows that the orbit $\mathcal{O}_M := \mathcal{O}_x$ of M has dimension

$$\dim \mathcal{O}_M = \dim GL(\mathbf{d}) - \dim \text{End}_{kQ}(M).$$

Lemma 3.1. ([10]) *Let Q be a Dynkin quiver, i.e., a disjoint union of oriented (simply-laced) Dynkin diagrams. For $x \in R(\mathbf{d}_1)$ and $y \in R(\mathbf{d}_2)$, let $\mathcal{E}(\mathcal{O}_x, \mathcal{O}_y)$ be the set of all $z \in R(\mathbf{d})$ where $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$ such that $V(z)$ is an extension of some $M \in \mathcal{O}_x$ by some $N \in \mathcal{O}_y$. Then $\mathcal{E}(\mathcal{O}_x, \mathcal{O}_y)$ is irreducible.*

Given representations M, N of Q , consider the extensions

$$0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$$

of M by N . By the lemma, there is a unique (up to isomorphism) such extension G with $\dim \mathcal{O}_G$ maximal (i.e., with $\dim \text{End}_{kQ}(G)$ minimal). We call G the *generic extension* of M by N , denoted by $M * N$.

For two representations M, N , we say that M degenerates to N , or that N is a degeneration of M , and write $[N] \leq [M]$ (or simply $N \leq M$), if $\mathcal{O}_N \subseteq \overline{\mathcal{O}_M}$, the closure of \mathcal{O}_M . Note that $N < M \iff \mathcal{O}_N \subseteq \overline{\mathcal{O}_M} \setminus \mathcal{O}_M$.

Remark 3.2. The relation \leq on the isoclasses is independent of the field k . This is seen from the following equivalence proved in [1, Prop. 3.2]:

$$N \leq M \iff \dim \text{Hom}(X, N) \geq \dim \text{Hom}(X, M), \forall X \quad (1)$$

and the fact that the dimension $\dim \text{Hom}(X, Y)$ is the same over any field. Thus, we may simply define a (characteristic-free) partial order on Λ by

$$\lambda \leq \mu \iff M_k(\lambda) \leq M_k(\mu).$$

for any given (algebraically closed) field k .

The first part of the following results is well-known (see, for example, [1, 1.1]) and the other parts are proved in [10].

Theorem 3.3. (1) *If $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$ is exact and non-split, then $M \oplus N < E$.*

(2) *Let M, N, X be representations of Q . Then $X \leq M * N$ if and only if there exist $M' \leq M, N' \leq N$ such that X is an extension of M' by N' . In particular, we have $M' \leq M, N' \leq N \implies M' * N' \leq M * N$.*

(3) *Let \mathcal{M} be the set of isoclasses of kQ -modules and define a multiplication $*$ on \mathcal{M} by $[M] * [N] = [M * N]$ for any $[M], [N] \in \mathcal{M}$. Then \mathcal{M} is a monoid with identity $1 = [0]$ and the multiplication $*$ preserves the induced partial ordering on \mathcal{M} .*

(4) *\mathcal{M} is generated by the simple modules $[S_i], i \in I$.*

Let Ω be the set of words on the alphabet $I = \{1, \dots, n\}$. For $w = i_1 i_2 \dots i_m \in \Omega$, let $\wp(w) \in \Lambda$ be the element defined by

$$[S_{i_1}] * \dots * [S_{i_m}] = [M(\wp(w))]. \quad (2)$$

Thus, we obtain a map $\wp : \Omega \rightarrow \Lambda$. Note that by the theorem above, \wp is surjective and induces a partition of $\Omega = \cup_{\lambda \in \Lambda} \Omega_\lambda$ with $\Omega_\lambda = \wp^{-1}(\lambda)$. Each Ω_λ is called a *fibre* of \wp .

Note from Remark 3.2 that, if we define $\lambda * \mu$ ($\lambda, \mu \in \Lambda$) by $M(\lambda * \mu) \cong M(\lambda) * M(\mu)$, then the element $\lambda * \mu$ is well-defined, independent of the field k . Note also that the multiplication $*$ on Λ depends on the orientation of Q .

4. THE POSET Λ

In this section we shall look at some properties of the poset (Λ, \leq) , where \leq is defined in Remark 3.2.

For $w = i_1 i_2 \cdots i_m \in \Omega$ and $\lambda \in \Lambda$, let φ_w^λ denote the Hall polynomial $\varphi_{\mu_1 \cdots \mu_m}^\lambda$, where $M(\mu_r) \cong S_{i_r}$. Thus, for a finite field k ,

$$\varphi_w^\lambda(q_k) = F_{S_{i_1 k} \cdots S_{i_m k}}^{M_k(\lambda)}$$

is the number of composition series of $M_k(\lambda)$:

$$M_k(\lambda) = M_0 \supset M_1 \supset \cdots \supset M_{m-1} \supset M_m = 0$$

with $M_{j-1}/M_j \cong S_{i_j k}$. Such a composition series is called a composition series of *type* w .

The following lemma is a bit stronger than [3, 6.2].

Lemma 4.1. *Let $w \in \Omega$ and $\mu \geq \lambda$ in Λ . Then $\varphi_w^\mu \neq 0$ implies $\varphi_w^\lambda \neq 0$.*

Proof. Let $w = i_1 i_2 \cdots i_m$ and $w' = i_2 \cdots i_m$. We apply induction on m . If $m = 1$ then $\mu \geq \lambda$ forces $M(\mu) = M(\lambda)$ and the result is clear. Assume now $m > 1$. If $\varphi_w^\mu \neq 0$, then $\varphi_w^\mu(q_k) \neq 0$ for some finite field k . Thus, $M_k(\mu)$ has a submodule $M'_k \cong M_k(\mu')$ which has a composition series of type w' . Hence, $\varphi_{w'}^{\mu'} \neq 0$ since $\varphi_{w'}^{\mu'}(q_k) \neq 0$. Base change to the algebraic closure \bar{k} of k gives an exact sequence over \bar{k} (We drop the subscripts \bar{k} .)

$$0 \longrightarrow M' \longrightarrow M(\mu) \longrightarrow S_{i_1} \longrightarrow 0.$$

Thus,

$$M(\lambda) \leq M(\mu) \leq S_{i_1} * M'.$$

By Theorem 3.3(2), there exist modules N', N'' such that $M(\lambda)$ is an extension of N' by N'' and $N' \leq M', N'' \leq S_{i_1}$. So we obtain an exact sequence (over \bar{k})

$$0 \longrightarrow N' \xrightarrow{f} M(\lambda) \xrightarrow{g} N'' \longrightarrow 0.$$

Now the condition $N' \leq M'$ means $\lambda' \leq \mu'$ where $N' \cong M(\lambda')$. Since $\varphi_{w'}^{\mu'} \neq 0$, it follows from induction that $\varphi_{w'}^{\lambda'} \neq 0$, that is, N' has a composition series of type w' . On the other hand, since S_{i_1} is simple, $N'' \leq S_{i_1}$ implies $N'' \cong S_{i_1}$. Therefore, $M(\lambda)$ has a composition series of type w , and consequently, $\varphi_w^\lambda \neq 0$. \square

We now relate the partial order \leq to certain non-zero Hall polynomials.

Theorem 4.2. *Let $\lambda, \mu \in \Lambda$. Then $\lambda \leq \mu$ if and only if there exists a word $w \in \wp^{-1}(\mu)$ with $\varphi_w^\lambda \neq 0$.*

Proof. Suppose $\lambda \leq \mu$. Since \wp is surjective, $\mu = \wp(w)$ for some $w \in \Omega$. By (2), we see that $\varphi_w^{\wp(w)} \neq 0$. Thus, Lemma 4.1 implies $\varphi_w^\lambda \neq 0$, as required.

Conversely, let $w = i_1 i_2 \cdots i_m \in \Omega$, $\lambda \in \Lambda$, and suppose $\varphi_w^\lambda \neq 0$. We use induction on m to prove that $\lambda \leq \wp(w)$. If $m = 1$, there is nothing to prove. Let $m > 1$ and $w' = i_2 \cdots i_m$ and assume $\lambda' \leq \wp(w')$ whenever $\varphi_{w'}^{\lambda'} \neq 0$. Since $\varphi_w^\lambda \neq 0$, there is a finite field k (of any given

characteristic) such that $\varphi_w^\lambda(q_k) \neq 0$. Thus, there is a submodule M'_k of $M_k(\lambda)$ which has a composition series of type w' . This implies $\varphi_{w'}^{\lambda'} \neq 0$ where $M_k(\lambda') \cong M'_k$. By induction, we have $\lambda' \leq \wp(w')$.

On the other hand, base change to the exact sequence

$$0 \longrightarrow M'_k \longrightarrow M_k(\lambda) \longrightarrow S_{i_1 k} \longrightarrow 0$$

yields an exact sequence over \bar{k}

$$0 \longrightarrow M' \longrightarrow M(\lambda) \longrightarrow S_{i_1} \longrightarrow 0.$$

(Here again we dropped the subscripts \bar{k} .) By Theorem 3.3(2) we obtain that

$$M(\lambda) \leq S_{i_1} * M(\lambda') \leq S_{i_1} * M(\wp(w')) = M(\wp(w)).$$

Therefore, $\lambda \leq \wp(w)$. □

5. DISTINGUISHED WORDS

Let $w = i_1 i_2 \cdots i_m$ be a word in Ω . Then w can be uniquely expressed in the *tight form* $w = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t}$, where $e_r \geq 1$, $1 \leq r \leq t$, and $j_r \neq j_{r+1}$ for $1 \leq r \leq t-1$. Following [13, 2.3], a filtration

$$M = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0$$

of a module is called a *reduced* filtration of type w if $M_{r-1}/M_r \cong e_r S_{j_r}$, for all $1 \leq r \leq t$. Note that any reduced filtration of M of type w can be refined to a composition series of M of type w . Conversely, given a composition series of M of type w , there is a unique reduced filtration of M of type w such that the given composition series is a refinement of this reduced filtration. By $\gamma_w^\lambda(q)$ we denote the Hall polynomial $\varphi_{\mu_1 \cdots \mu_t}^\lambda(q)$, where $M(\mu_r) = e_r S_{j_r}$. Thus, for any finite field k of q_k elements, $\gamma_w^\lambda(q_k)$ is the number of the reduced filtrations of $M_k(\lambda)$ of type w . A word w is called *distinguished* if $\gamma_w^{\wp(w)}(q) = 1$. Note that w is distinguished if and only if, for some algebraically closed field k , $M_k(\wp(w))$ has a unique reduced filtration of type w (cf. [3, §5]).

Example 5.1. Let $w = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t}$ be in the tight form. If j_1, \dots, j_t are pairwise distinct and satisfy

$$\text{Ext}_{kQ}^1(S_{j_r}, S_{j_s}) \neq 0 \implies r < s,$$

then $F_{N_1 \cdots N_t}^M = 0$ or 1 for every kQ -module M , where $N_r = e_r S_{j_r}$. Thus, w is distinguished.

Distinguished words will be used in the construction of integral monomial bases for the Lusztig form. The following lemma shows that these words are somehow evenly distributed.

Lemma 5.2. *Each fibre of \wp contains at least one distinguished word.*

Proof. This follows directly from [11, Lemma 4.5]. For completeness, we present here the construction of such distinguished words.

By \mathcal{I} we denote the set of the isoclasses of indecomposable representations of Q . Let \mathcal{I}_* be a *directed* partition of \mathcal{I} (see [11, §4]), that is, a partition of the set \mathcal{I} into subsets $\mathcal{I}_1, \dots, \mathcal{I}_m$ such that

- a) $\text{Ext}_{kQ}^1(M, N) = 0$ for all M, N in the same part \mathcal{I}_r ,
- b) $\text{Ext}_{kQ}^1(M, N) = 0 = \text{Hom}_{kQ}(N, M)$ if $M \in \mathcal{I}_r$, $N \in \mathcal{I}_s$, where $1 \leq r < s \leq m$.

Then, for each $\lambda \in \Lambda$, we have a unique decomposition

$$M(\lambda) = \bigoplus_{r=1}^m M_r,$$

where all the summands of M_r belong to \mathcal{I}_r , $1 \leq r \leq m$. Thus,

$$\mathrm{Hom}_{kQ}(M_r, M_s) \neq 0 \implies r \leq s. \quad (3)$$

Further, we order the vertices of Q in a sequence i_1, i_2, \dots, i_n such that, for each $1 < j \leq n$, i_j is either a sink or an isolated vertex in the full subquiver of Q with vertices $\{i_1, \dots, i_{j-1}, i_j\}$. Equivalently, i_1, i_2, \dots, i_n are ordered to satisfy

$$\mathrm{Ext}_{kQ}^1(S_{i_j}, S_{i_l}) \neq 0 \implies j < l. \quad (4)$$

Let $\mathbf{d}^{(r)} = (d_1^{(r)}, \dots, d_n^{(r)}) = \dim M_r$, $1 \leq r \leq m$, and set

$$w_r = \underbrace{i_1 \cdots i_1}_{d_{i_1}^{(r)}} \cdots \underbrace{i_n \cdots i_n}_{d_{i_n}^{(r)}}$$

and $w_\lambda = w_1 \cdots w_m \in \Omega$. Then [11, Lemma 4.5] implies that $\wp(w_\lambda) = \lambda$ and $\gamma_{w_\lambda}^\lambda(q) = 1$, that is, w_λ is distinguished. \square

We call the distinguished words constructed above *directed* distinguished words (with respect to the given directed partition \mathcal{I}_*).

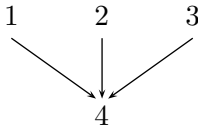
We mention a special case of directed partitions \mathcal{I}_* where each part \mathcal{I}_r contains only one isoclass. This case is equivalent to ordering the indecomposable modules $M(\beta_1), M(\beta_2), \dots, M(\beta_\nu)$ such that

$$\mathrm{Hom}_{kQ}(M(\beta_r), M(\beta_s)) \neq 0 \implies r \leq s. \quad (5)$$

Note that monomial bases associated to these special directed distinguished words have been constructed in [9] and [15]; see Remarks 6.5 below.

The following example shows that a fibre of \wp could contain many words other than directed distinguished ones.

Example 5.3. Let Q denote the following quiver



Let $\lambda \in \Lambda$ be such that $M(\lambda)$ is the indecomposable kQ -module with dimension vector $(1, 1, 1, 2)$. Then $\wp^{-1}(\lambda)$ contains 12 words

$$\begin{aligned} &1234^2, 1324^2, 2134^2, 2314^2, 3124^2, 3214^2 \\ &12434, 13424, 21434, 23414, 31424, 32414 \end{aligned}$$

which are all distinguished. From the structure of the Auslander-Reiten quiver of kQ , one sees easily that the first 6 words are directed distinguished, but the last 6 words are not.

6. MONOMIAL AND INTEGRAL MONOMIAL BASES

For $m \geq 1$, let $\llbracket m \rrbracket^! = \llbracket 1 \rrbracket \llbracket 2 \rrbracket \cdots \llbracket m \rrbracket$ where $\llbracket e \rrbracket = \frac{q^e - 1}{q - 1}$. Then $\llbracket m \rrbracket = v^{m-1}[m]$ and $\llbracket m \rrbracket^! = v^{\frac{m(m-1)}{2}}[m]^!$.

Lemma 6.1. *Let $w \in \Omega$ be a word with the tight form $j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t}$. Then, for each $\lambda \in \Lambda$,*

$$\varphi_w^\lambda(q) = \gamma_w^\lambda(q) \prod_{r=1}^t \llbracket e_r \rrbracket^!.$$

In particular, $\varphi_w^{\wp(w)} = \prod_{r=1}^t \llbracket e_r \rrbracket^!$ if w is distinguished.

Proof. The result follows from the fact that the number of composition series of eS_i is $\llbracket e \rrbracket^!$ (cf. [14, 8.2]) and the definition of a distinguished word. \square

To each word $w = i_1 i_2 \cdots i_m \in \Omega$, we associate a monomial

$$u_w = u_{i_1} u_{i_2} \cdots u_{i_m} \in \mathcal{H}.$$

Theorem 4.2 and Lemma 6.1 give the following.

Proposition 6.2. *For each $w \in \Omega$ with the tight form $j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t}$, we have*

$$u_w = \sum_{\lambda \leq \wp(w)} \varphi_w^\lambda(q) u_\lambda = \prod_{r=1}^t \llbracket e_r \rrbracket^! \sum_{\lambda \leq \wp(w)} \gamma_w^\lambda(q) u_\lambda. \quad (6)$$

Moreover, the coefficients appearing in the sum are all non-zero.

We remark that this result improves [15, Thm 1, p.96] in two aspects. First, we generalize the formula from certain directed distinguished words to all words; second, we replace the lexicographical order used in [15] by the Bruhat type partial order \leq .

For any commutative ring \mathcal{A}' which is an \mathcal{A} -algebra and any \mathcal{A} -module M , let $M_{\mathcal{A}'} = \mathcal{A}' \otimes_{\mathcal{A}} M$ denote the \mathcal{A}' -module obtained from M by base change to \mathcal{A}' .

Theorem 6.3. *For every $\lambda \in \Lambda$, choose an arbitrary word $w_\lambda \in \wp^{-1}(\lambda)$. The set $\{u_{w_\lambda} | \lambda \in \Lambda\}$ is a $\mathbb{Q}(q)$ -basis of $\mathcal{H}_{\mathbb{Q}(q)}$. Moreover, if all w_λ are chosen to be distinguished, then this set is an $\mathcal{A}_{(q-1)}$ -basis of $\mathcal{H}_{\mathcal{A}_{(q-1)}}$ where $\mathcal{A}_{(q-1)}$ denotes the localization of \mathcal{A} at the maximal ideal generated by $q - 1$.*

Proof. The theorem follows from Prop. 5.3 and the fact that $\varphi_{w_\lambda}^{\wp(w_\lambda)}$ is invertible in $\mathcal{A}_{(q-1)}$ if w_λ is distinguished. \square

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the Lie algebra over \mathbb{Q} of type Q with generators e_i, f_i, h_i . Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of \mathfrak{g} . We also define monomials e_w similarly for $w \in \Omega$ in $\mathfrak{U}(\mathfrak{n}_+)$. Then, we have the following.

Corollary 6.4. *For every $\lambda \in \Lambda$, choose an arbitrary distinguished word $w_\lambda \in \wp^{-1}(\lambda)$. The set $\{e_{w_\lambda} | \lambda \in \Lambda\}$ is a \mathbb{Q} -basis of $\mathfrak{U}(\mathfrak{n}_+)$.*

Proof. The result follows from the isomorphism $\mathcal{H}_{\mathcal{A}'} / (q - 1)\mathcal{H}_{\mathcal{A}'} \cong \mathfrak{U}(\mathfrak{n}_+)$, where $\mathcal{A}' = \mathcal{A}_{(q-1)}$, and Theorem 6.3. \square

Proof of Theorem 1.1. We first observe that, for each $w = i_1 i_2 \cdots i_m \in \Omega$,

$$u_{i_1} \star \cdots \star u_{i_m} = v^{\varepsilon(w)} u_w$$

where

$$\varepsilon(w) = \sum_{1 \leq r < s \leq m} \langle \dim S_{i_r}, \dim S_{i_s} \rangle.$$

Let, for $w = j_1^{e_1} \cdots j_t^{e_t}$ in the tight form,

$$\mathbf{m}^{(w)} := E_{j_1}^{(e_1)} \cdots E_{j_t}^{(e_t)} = \left(\prod_{r=1}^t [e_r]! \right)^{-1} u_{j_1}^{\star e_1} \star \cdots \star u_{j_t}^{\star e_t}.$$

Since $\prod_{r=1}^t [e_r]! = v^{-\delta(w)} \prod_{r=1}^t \llbracket e_r \rrbracket!$, where $\delta(w) = \sum_{r=1}^t \frac{e_r(e_r-1)}{2}$, it follows from Prop. 6.2 that

$$\mathbf{m}^{(w)} = \left(\prod_{r=1}^t \llbracket e_r \rrbracket! \right)^{-1} v^{\delta(w)+\varepsilon(w)} u_w = v^{\delta(w)+\varepsilon(w)} \sum_{\lambda \leq \wp(w)} \gamma_w^\lambda(v^2) u_\lambda. \quad (7)$$

This together with Prop. 2.1 and Theorem 6.3 implies Theorem 1.1 with $\Omega_\lambda = \wp^{-1}(\lambda)$ for all $\lambda \in \Lambda$.

Remarks 6.5. (a) It is clear, from the definition, that the monomial basis $\{E^{(M)}\}$ constructed in [11, Thm 4.2] involves only directed distinguished words w_λ .

(b) As a special case of [11, Thm 4.2], the monomial bases constructed in [9, 7.8] and [15, pp.101–2] involve only those directed distinguished words defined with respect to the special directed partition \mathcal{I}_* satisfying the conditions (5) and (4) in the previous section; see [15, Thm 1] and [9, 4.12(c), 4.13]².

We now briefly look at the elementary and algebraic construction of the canonical basis for U^+ (cf. [10, §6]). It should be pointed out that the elementary constructions given in, e.g., [9], [6], [15] and [2] used a finer order than the one used in the geometric construction. We now use the same order which has an algebraic interpretation (1).

For each $\lambda \in \Lambda$, let

$$\tilde{u}_\lambda = v^{-\dim M(\lambda) + \dim \text{End}(M(\lambda))} u_\lambda.$$

Then, by Prop. 2.1, U^+ is \mathcal{Z} -free with basis $\mathcal{E} = \{\tilde{u}_\lambda : \lambda \in \Lambda\}$. Note that $U^+ = \oplus_{\mathbf{d}} U_{\mathbf{d}}^+$ is $\mathbb{N}I$ -graded according to the dimension vectors, and each $U_{\mathbf{d}}^+$ is \mathcal{Z} -free with basis $\mathcal{E} \cap U_{\mathbf{d}}^+ = \{\tilde{u}_\lambda : \lambda \in \Lambda_{\mathbf{d}}\}$. Clearly, each $\Lambda_{\mathbf{d}}$ together with \leq is a poset.

Define $\iota : U^+ \rightarrow U^+$ by setting $\iota(E_i^{(m)}) = E_i^{(m)}$ and $\iota(v) = v^{-1}$. Clearly, ι preserves the grading of U^+ . Write for any $\tilde{u}_\lambda \in U_{\mathbf{d}}^+$

$$\iota(\tilde{u}_\mu) = \sum_{\lambda} r_{\lambda, \mu} \tilde{u}_\lambda. \quad (8)$$

By [9, 9.10] (see [4] for more details), the existence of the canonical bases for $U_{\mathbf{d}}^+$ follows from the following property on the coefficients $r_{\lambda, \mu}$

$$r_{\lambda, \lambda} = 1, r_{\lambda, \mu} = 0 \text{ unless } \lambda \leq \mu. \quad (9)$$

²It seems to us that the condition (4) was not explicitly stated in [9], but was implicitly used in [9, 7.2].

We will use (7) to derive (9). We first calculate $\delta(w) + \varepsilon(w)$ for directed distinguished words (cf. [15, Lemma, p.102]).

Lemma 6.6. *We have for any directed distinguished word $w \in \Omega$*

$$\delta(w) + \varepsilon(w) = -\dim M(\wp(w)) + \dim \text{End } M(\wp(w)).$$

Proof. Let $w \in \Omega$ be a directed distinguished word. Then, by definition, there is a directed partition \mathcal{I}_* of \mathcal{I} and a $\lambda \in \Lambda$ such that w has the form $w = w_\lambda = w_1 \cdots w_m$ with

$$w_r = \underbrace{i_1 \cdots i_1}_{d_{i_1}^{(r)}} \cdots \underbrace{i_n \cdots i_n}_{d_{i_n}^{(r)}},$$

where $M(\lambda) = M_1 \oplus M_2 \oplus \cdots \oplus M_m$, $\mathbf{d}^{(r)} = (d_1^{(r)}, \dots, d_n^{(r)}) = \mathbf{dim } M_r$, $1 \leq r \leq m$, and the sequence i_1, i_2, \dots, i_n of vertices are ordered to satisfy (4). Clearly, we have

$$\delta(w) = \sum_{r=1}^m \sum_{j=1}^n \frac{d_{i_j}^{(r)}(d_{i_j}^{(r)} - 1)}{2}.$$

Since $\langle \mathbf{dim } S_{i_j}, \mathbf{dim } S_{i_l} \rangle = 0$ for $j > l$ and $\text{Ext}^1(M_r, M_s) = 0$ for all $1 \leq r \leq s \leq m$, we obtain, for each $1 \leq r \leq m$,

$$\begin{aligned} \varepsilon(w_r) &= \sum_{j=1}^n \frac{d_{i_j}^{(r)}(d_{i_j}^{(r)} - 1)}{2} \langle \mathbf{dim } S_{i_j}, \mathbf{dim } S_{i_j} \rangle \\ &\quad + \sum_{1 \leq j < l \leq n} \langle \mathbf{dim } d_{i_j}^{(r)} S_{i_j}, \mathbf{dim } d_{i_l}^{(r)} S_{i_l} \rangle \\ &= \langle \mathbf{dim } M_r, \mathbf{dim } M_r \rangle - \sum_{j=1}^n \frac{(d_{i_j}^{(r)})^2}{2} - \sum_{j=1}^n \frac{d_{i_j}^{(r)}}{2} \\ &= \dim \text{End}(M_r) - \sum_{j=1}^n \frac{d_{i_j}^{(r)}(d_{i_j}^{(r)} + 1)}{2} \end{aligned}$$

and therefore,

$$\begin{aligned} \varepsilon(w) &= \sum_{r=1}^m \varepsilon(w_r) + \sum_{1 \leq r < s \leq m} \langle \mathbf{dim } M_r, \mathbf{dim } M_s \rangle \\ &= \sum_{r=1}^m \varepsilon(w_r) + \sum_{1 \leq r < s \leq m} \dim \text{Hom}(M_r, M_s). \end{aligned}$$

Noting $\text{Hom}(M_r, M_s) = 0$ for $r > s$, we finally obtain

$$\begin{aligned} &\delta(w) + \varepsilon(w) \\ &= \sum_{r=1}^m \dim \text{End}(M_r) + \sum_{1 \leq r < s \leq m} \dim \text{Hom}(M_r, M_s) - \sum_{r=1}^m \sum_{j=1}^n d_{i_j}^{(r)} \\ &= \dim \text{End}(M(\lambda)) - \dim M(\lambda). \end{aligned}$$

This completes the proof. \square

Remark 6.7. It would be interesting to know if the lemma holds for all distinguished words.

By the lemma and (7), we have for a directed distinguished word w

$$\mathbf{m}^{(w)} = \tilde{\mathbf{u}}_{\wp(w)} + \sum_{\lambda < \wp(w)} f_{\lambda, \wp(w)} \tilde{\mathbf{u}}_{\lambda}, \quad (10)$$

where $0 \neq f_{\lambda, \wp(w)} \in \mathcal{Z}$. If we fix a representative set $\Lambda' = \{w_{\lambda} : \lambda \in \Lambda\}$, where $w_{\lambda} \in \Omega_{\lambda}$, consisting of directed distinguished words, then the above relation implies that, for any $\lambda \in \Lambda$,

$$\tilde{\mathbf{u}}_{\lambda} \in \mathbf{m}^{(w_{\lambda})} + \sum_{\mu < \lambda} \mathcal{Z} \mathbf{m}^{(w_{\mu})}.$$

Restricting to $\Lambda_{\mathbf{d}}$ where \mathbf{d} is a fixed dimension vector, we obtain the transition matrix $(f_{\lambda, \mu})_{\lambda, \mu \in \Lambda_{\mathbf{d}}}$. This matrix has an inverse $(g_{\lambda, \mu})_{\lambda, \mu \in \Lambda_{\mathbf{d}}}$ satisfying $g_{\lambda, \lambda} = 1$ and $g_{\lambda, \mu} = 0$ unless $\lambda \leq \mu$. Thus we have

$$\tilde{\mathbf{u}}_{\mu} = \mathbf{m}^{(w_{\mu})} + \sum_{\lambda < \mu} g_{\lambda, \mu} \mathbf{m}^{(w_{\lambda})}.$$

Applying ι , we obtain by (10)

$$\iota(\tilde{\mathbf{u}}_{\mu}) = \mathbf{m}^{(w_{\mu})} + \sum_{\lambda < \mu} \bar{g}_{\lambda, \mu} \mathbf{m}^{(w_{\lambda})} = \tilde{\mathbf{u}}_{\mu} + \sum_{\lambda < \mu} r_{\lambda, \mu} \tilde{\mathbf{u}}_{\lambda}. \quad (11)$$

This proves that the coefficients in (8) satisfy (9). Thus, the corresponding canonical basis $\{\mathbf{c}_{\lambda}\}_{\lambda \in \Lambda}$ is uniquely defined.

Remarks 6.8. (a) The canonical basis defined above is the same as Lusztig's canonical basis. This is because the basis \mathcal{E} is a PBW type basis (see [16, Thm 7]). We also note that, as in the Hecke algebra case ([7], [8]), the partial order used in this construction is the same as the one used in the geometric construction (see [9, §9]).

(b) We may also use non-directed distinguished words in the construction. Though (10) needs to be adjusted by a power of v , the relation (11) will remain the same, and hence, the canonical basis will be the same.

7. EXAMPLE: THE TYPE A CASE

In this section, we shall give a combinatorial description of the map $\wp : \Omega \rightarrow \Lambda$ for the following linear quiver:

$$Q = A_n : 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n.$$

We will also give an explicit description of the distinguished words in this case. Since A_n is a subquiver of a cyclic quiver, results obtained below and their proofs are similar to (or even simpler than) those given in [3] and will be mostly omitted.

It is known that, for $1 \leq i \leq j \leq n$, there is a unique (up to isomorphism) indecomposable kA_n -module M_{ij} with top S_i and of length $j - i + 1$, and all M_{ij} , $1 \leq i \leq j \leq n$, form a complete set of non-isomorphic indecomposable kA_n -modules. By Gabriel's theorem, each M_{ij} corresponds to a positive root β_{ij} . Thus, we have $\Phi^+ = \{\beta_{ij} | 1 \leq i \leq j \leq n\}$. For each map $\lambda \in \Lambda$, we set $\lambda_{ij} = \lambda(\beta_{ij})$. First, we have the following positivity result which can be proved by counting and induction on the length of w (cf. [3, Prop. 9.1]).

Proposition 7.1. *For each $w \in \Omega$ and each $\lambda \in \Lambda$, the polynomial φ_w^λ lies in $\mathbb{N}[q]$.*

Now, for each $i \in I$, we define a map $\sigma_i : \Lambda \rightarrow \Lambda$ as follows. For $\lambda \in \Lambda$, if S_{i+1} is not a summand of $M(\lambda)/\text{rad}M(\lambda)$ (i.e., $\lambda_{i+1,l} = 0, \forall l$), then $\sigma_i \lambda$ is obtained by adding 1 to λ_{ii} so that $M(\sigma_i \lambda) = M(\lambda) \oplus S_i$; otherwise, $\sigma_i \lambda$ is defined by

$$(\sigma_i \lambda)_{rs} = \begin{cases} \lambda_{rs} & \text{if } (r, s) \neq (i, j), (i+1, j) \\ \lambda_{ij} + 1 & \text{if } (r, s) = (i, j) \\ \lambda_{i+1,j} - 1 & \text{if } (r, s) = (i+1, j), \end{cases}$$

where j is the maximal index with $\lambda_{i+1,j} \neq 0$. We have the following (cf. [3, Prop. 3.7]).

Proposition 7.2. *Let $i \in I$ and $\lambda \in \Lambda$. Then*

$$S_i * M(\lambda) \cong M(\sigma_i \lambda).$$

Therefore, for any $w = i_1 \cdots i_m \in \Omega$, $\wp(w) = \sigma_{i_1} \cdots \sigma_{i_m}(0)$.

Let $w = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \Omega$ be in the tight form. For each $0 \leq r \leq t$, we put $w_r = j_{r+1}^{e_{r+1}} \cdots j_t^{e_t}$ and $\lambda^{(r)} = \wp(w_r)$. In particular, $w_0 = w$ and $w_t = 1$. Further, for $r \geq 1$, we have

$$\lambda^{(r-1)} = \wp(w_{r-1}) = \underbrace{\sigma_{j_r} \cdots \sigma_{j_r}}_{e_r}(\lambda^{(r)}).$$

The following result gives a combinatorial description of distinguished words (cf. [3, 5.5]).

Proposition 7.3. *Let $w = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \Omega$ and $\lambda^{(r)}, 0 \leq r \leq t$, be given as above. Then w is distinguished if and only if, for each $1 \leq r \leq t$, either $\lambda_{j_r j}^{(r)} = 0$ for all $j_r \leq j \leq n$, or $e_r \leq \sum_{a=l_r+1}^n \lambda_{j_r+1,a}^{(r)}$ where l_r is the maximal index for which $\lambda_{j_r l_r}^{(r)} \neq 0$.*

Proof. Using a similar argument as in [3, Thm 5.5], one can show that w is distinguished if and only if, for each $1 \leq r \leq t$, $M(\lambda^{(r-1)})$ admits a unique submodule isomorphic to $M(\lambda^{(r)})$. However, the latter condition is equivalent to the described combinatorial condition, as shown in [3, Lemma 5.4]. \square

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